

# The Jacobian Conjecture, a Reduction of the Degree to the Quadratic Case

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## Abstract

The Jacobian Conjecture states that any locally invertible polynomial system in  $\mathbb{C}^n$  is globally invertible with polynomial inverse. C. W. Bass *et. al.* (1982) proved a reduction theorem stating that the conjecture is true for any degree of the polynomial system if it is true in degree three. This degree reduction is obtained with the price of increasing the dimension  $n$ .

We prove here a theorem concerning partial elimination of variables, which implies a reduction of the generic case to the quadratic one. The price to pay is the introduction of a supplementary parameter  $0 \leq n' \leq n$ , parameter which represents the dimension of a linear subspace where some particular conditions on the system must hold.

We first give a purely algebraic proof of this reduction result and we then expose a distinct proof, in a Quantum Field Theoretical formulation, using the *intermediate field method*.

## 1 Introduction

The *Jacobian conjecture* has been formulated in [Kel39], as a strikingly simple and natural conjecture concerning the global invertibility of polynomial systems. Later on, it also appeared connected to questions in non-commutative algebra, in particular the conjecture has been shown to be stably equivalent to the Dixmier Conjecture (see [BKK07]), which concerns endomorphisms of the Weyl algebra. Despite several efforts, and various promising partial results, it remains unsolved. An introduction to the problem, the context, and the state of the art up to 1982, can be found in the paper [BCW82],

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which provides both a clear review, and among the most relevant advances on the problem.

The function  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is said to be a *polynomial system* if all the coordinate functions  $F_j$ 's are polynomials. Let us call  $\mathcal{P}_n$  the set of such functions. For a function  $F$ , define

$$J_F(z) = \left( \frac{d}{dz_i} F_j(z) \right)_{1 \leq i, j \leq n}, \quad (1)$$

the corresponding Jacobian matrix. Then  $\det J_F(z)$  is itself a polynomial, and it is nowhere zero iff it is a non-zero constant. Local invertibility in  $y = F(z)$  is related to the invertibility of  $J_F(z)$  as a matrix, and thus to the vanishing of its determinant. Depending from the space of functions under analysis, local invertibility may or may not be sufficient to imply global invertibility. The question here is what is the case, for the space  $\mathcal{P}_n$  of polynomial systems over an algebraically-closed field of characteristic zero (such as  $\mathbb{C}$ ). For this reason, we define the two subspaces of  $\mathcal{P}_n$

**Definition 1.1.**

$$\mathcal{J}_n^{\text{lin}} := \{F \in \mathcal{P}_n \mid \det J_F(z) = c \in \mathbb{C}^\times\}, \quad (2)$$

$$\mathcal{J}_n := \{F \in \mathcal{P}_n \mid F \text{ is invertible}\}. \quad (3)$$

For the questions at hand, it will often be sufficient to analyse the subset of  $\mathcal{J}_n^{\text{lin}}$  such that  $\det J_F(z) = 1$ .

More precisely w.r.t. what anticipated above, one can see, e.g., from [BCW82, Thm. 2.1, pag. 294], that  $F \in \mathcal{J}_n^{\text{lin}}$  is a necessary condition for the invertibility of  $F$ , and, if  $F$  is invertible, the (set theoretic) inverse is automatically polynomial and unique. The question is whether the condition on the Jacobian is also sufficient, i.e.

**Conjecture 1.2** (Jacobian Conjecture, [Kel39]).

$$\mathcal{J}_n^{\text{lin}} = \mathcal{J}_n \quad \forall n. \quad (4)$$

Define the *total degree* of  $F$ ,  $\deg(F)$ , as  $\max_j \deg(F_j(z))$ , and introduce the subspaces

$$\mathcal{P}_{n,d} = \{F \in \mathcal{P}_n \mid \deg(F) \leq d\}; \quad (5)$$

and similarly for  $\mathcal{J}$  and  $\mathcal{J}^{\text{lin}}$ . We mention two positive results on the conjecture. First, a theorem for the quadratic case ( $d = 2$ ), established first in [Wan80], and then, through a much simpler proof, in [Oda80] (see also [Wri89, Lemma 3.5] and [BCW82, Thm. 2.4, pag. 298]).

**Theorem 1.3** ([Wan80]).

$$\mathcal{J}_{n,2}^{\text{lin}} = \mathcal{J}_{n,2} \quad \forall n. \quad (6)$$

Then, a reduction theorem, from the general case to the cubic case, established by Bass, Connell and Wright [BCW82, Sec. II]).

**Theorem 1.4** ([BCW82]).

$$\mathcal{J}_{n,3}^{\text{lin}} = \mathcal{J}_{n,3} \quad \forall n \quad \implies \quad \mathcal{J}_n^{\text{lin}} = \mathcal{J}_n \quad \forall n. \quad (7)$$

The proof of the above theorem involves manipulations under which the dimension  $n$  of the system is increased, thus this proof does *not* imply the corresponding statement without the “ $\forall n$ ” quantifier, i.e. that  $\mathcal{J}_{n,3}^{\text{lin}} = \mathcal{J}_{n,3} \Rightarrow \mathcal{J}_n^{\text{lin}} = \mathcal{J}_n$ .

Our result is a reduction theorem, in the line of the one above, that tries to ‘fill the gap’ between the cases of degree two and three. However, for this, we have to introduce an adaptation of the statement of the Jacobian conjecture with one more parameter.

For  $n' \leq n$  and  $F \in \mathcal{P}_{n,d}$ , we write  $z = (z_1, z_2)$  and  $F = (F_1, F_2)$  to distinguish components in the two subspaces  $\mathbb{C}^{n'} \times \mathbb{C}^{n-n'} \equiv \mathbb{C}^n$ . We set  $R(z_2; z_1)$  to be equal to  $F_2(z_1, z_2)$ , emphasizing that, in  $R$ , we consider  $z_2$  as the variables in a polynomial system, and  $z_1$  as parameters. The invertibility of  $R$ , denoted by  $R(\cdot; z_1) \in \mathcal{J}_{n-n',d}$ , for a fixed  $z_1$ , means that there exists a polynomial  $R^{-1}$  with variables  $y_2 \in \mathbb{C}^{n-n'}$ , and depending on  $z_1$ , such that

$$\forall z_2 \in \mathbb{C}^{n-n'}, \quad R^{-1}(R(z_2; z_1); z_1) = z_2.$$

We are now ready to define the main objects involved in the reduction theorem of this paper.

**Definition 1.5.** *We define the subspaces of  $\mathcal{P}_{n,d}$*

$$\begin{aligned} \mathcal{J}_{n,d;n'} &:= \{F \in \mathcal{P}_{n,d} \mid R(\cdot; z_1) \in \mathcal{J}_{n-n',d} \quad \forall z_1 \in \mathbb{C}^{n'} \\ &\quad \text{and } F^{-1} \text{ restricted to } \mathbb{C}^{n'} \times \{0\} \text{ is in } \mathcal{P}_{n'}\} \\ \mathcal{J}_{n,d;n'}^{\text{lin}} &:= \{F \in \mathcal{P}_{n,d} \mid R(\cdot; z_1) \in \mathcal{J}_{n-n',d} \quad \forall z_1 \in \mathbb{C}^{n'} \\ &\quad \text{and } (\det J_F)(z_1, R^{-1}(0, z_1)) = c \in \mathbb{C}^\times, \quad \forall z_1 \in \mathbb{C}^{n'}\} \end{aligned}$$

The first definition is a clear and natural generalisation of  $\mathcal{J}_{n,d}$ , which corresponds to  $n' = n$ . The second one evaluates the Jacobian on a suitable algebraic variety, candidate image of  $\mathbb{C}^{n'} \times \{0\}$  under  $F^{-1}$  (the reason for this choice will become clear only in the following, see in particular Section

4). As a consequence, we have that  $\mathcal{J}_{n,d;n'}^{\text{lin}} \subseteq \mathcal{J}_{n,d;n'}$ , similarly to the original  $n' = n$  case.

Apparently, the most natural and intrinsic generalization would have been to consider an arbitrary linear subspace of dimension  $n - n'$ , on which  $z$  shall vanish, instead of the last  $n - n'$  variables. However, it will be notationally simpler to restrict to our choice, and cause no loss of generality for the problem at hand, which is clearly  $\text{GL}(n, \mathbb{C})$ -invariant.

Let us now state our reduction theorem:

**Theorem 1.6.** *For  $n \in \mathbb{N}$  and  $d \geq 3$ , there exists an injective map  $\Phi : \mathcal{P}_{n,d} \rightarrow \mathcal{P}_{n(n+1),d-1}$  satisfying*

$$\Phi(\mathcal{J}_{n,d}^{\text{lin}}) \equiv \mathcal{J}_{n(n+1),d-1;n}^{\text{lin}} \cap \text{Im}(\Phi); \quad \Phi(\mathcal{J}_{n,d}) \equiv \mathcal{J}_{n(n+1),d-1;n} \cap \text{Im}(\Phi), \quad (8)$$

where  $\text{Im}(\Phi) = \Phi(\mathcal{P}_{n,d})$ .

Combining Theorem 1.4 and the theorem above, the full Jacobian Conjecture reduces to the question whether

$$\mathcal{J}_{n(n+1),2;n}^{\text{lin}} \cap \text{Im}(\Phi) = \mathcal{J}_{n(n+1),2;n} \cap \text{Im}(\Phi).$$

This question seems at a first stage as difficult as the original Jacobian conjecture. However, it involves only a quadratic degree, and this might simplify the resolution, in the light of Wang Theorem (Thm. 1.3).

It is also natural, at this point, to formulate a stronger version of the Jacobian Conjecture

**Conjecture 1.7.** *For all  $n \geq n' \geq 0$ , and all  $d \geq 1$ ,*

$$\mathcal{J}_{n,d;n'}^{\text{lin}} = \mathcal{J}_{n,d;n'}. \quad (9)$$

As we have seen, the original Conjecture 1.2 follows from the cases  $(n, d; n') \in \{(m(m+1), 2, m)\}_{m \geq 0}$  of the above conjecture, restricted to  $\text{Im}(\Phi)$ .

As already announced above, this paper proves Theorem 1.6. We found this restriction of degree originally in the Quantum Field Theory formulation of the Jacobian conjecture, by applying the intermediate field method. So, in this paper, we first prove the theorem algebraically in section 2 and then, we expose the equivalent proof using combinatorial Quantum Field Theory (QFT) in subsection 3.3. For general references on QFT for combinatorists, we refer the interested reader to [Abd03b] or [Tan12], while a rederivation of the QFT analogs of quantities pertinent here is done in subsection 3.1, at a heuristic level, and in subsection 3.2, more formally.

For the sake of completeness, let us recall for the interested reader that various purely combinatorial approaches to the Jacobian Conjecture were given. Thus, the first paper of this series was the one of Zeilberger [Zei87], which proposes the Joyal method of combinatorial species as an appropriate tool to tackle the conjecture. His work has been followed by the one of Wright's [Wri99], which used trees to reformulate the conjecture and then by the one of Singer [Sin01], which used rooted trees (see also [Wri06] and [Sin11]).

## 2 Algebraic proof of the reduction theorem

In this section we use algebraic methods to prove Theorem 1.6. Let us first prove the following lemmas.

**Lemma 2.1** (Partial elimination, linearized version). *Let  $N = n_1 + n_2$ , and  $S \in \mathcal{P}_N$ . Write  $z = (z_1, z_2)$  for  $z \in \mathbb{K}^N = \mathbb{K}^{n_1} \times \mathbb{K}^{n_2}$  and so on. Call  $R(z_2; z_1) = S_2(z_1, z_2)$  the system in  $\mathcal{P}_{n_2}$ , where  $z_1$  coordinates are intended as parameters. Assume by hypothesis that  $R(\cdot; z_1) \in \mathcal{J}_{n_2}$  for all  $z_1 \in \mathbb{K}^{n_1}$ . Define  $H(z_1; y_2) = S_1(z_1, R^{-1}(y_2; z_1)) \in \mathcal{P}_{n_1}$ . We have*

$$S \in \mathcal{J}_{N; n_1}^{\text{lin}} \quad \text{iff} \quad H(\cdot; 0) \in \mathcal{J}_{n_1}^{\text{lin}}. \quad (10)$$

*Proof.* We actually prove that

$$\det J_S(z_1, R^{-1}(y_2; z_1)) = (\det J_{R(\cdot; z_1)}) \det J_{H(\cdot; y_2)}(z_1),$$

while  $\det J_{R(\cdot; z_1)} \in \mathbb{K}^\times$  is fixed by hypothesis.

Let us start by calculating explicitly  $\det J_S(z)$ . Expressing  $S(z)$  in terms of  $S_{1,2}$  and  $z_{1,2}$  gives the block decomposition

$$\frac{d}{dz} S(z) = \left( \begin{array}{c|c} \frac{d}{dz_1} S_1(z_1, z_2) & \frac{d}{dz_1} S_2(z_1, z_2) \\ \hline \frac{d}{dz_2} S_1(z_1, z_2) & \frac{d}{dz_2} S_2(z_1, z_2) \end{array} \right). \quad (11)$$

We express the determinant of the matrix above by mean of the Schur complement formula<sup>1</sup>. Recognize that  $\frac{d}{dz_2} S_2(z_1, z_2) = \frac{d}{dz_2} R(z_2; z_1) = J_{R(\cdot; z_1)}$ . Thus

$$\det \left( \frac{dS(z)}{dz} \right) = (\det J_{R(\cdot; z_1)}) \det \left( \frac{dS_1(z_1, z_2)}{dz_1} - \frac{dS_2(z_1, z_2)}{dz_1} (J_{R(\cdot; z_1)})^{-1} \frac{dS_1(z_1, z_2)}{dz_2} \right).$$

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<sup>1</sup>For a block matrix  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ , the *Schur complement formula* states  $\det M = (\det D) \det(A - BD^{-1}C)$ .

Recall that  $z_2 = R^{-1}(y_2; z_1)$ , however, if one wants to substitute one for the other in the expression above, some care is required in how to interpret derivatives w.r.t.  $z_1$  and  $z_2$ . We pass instead to the second evaluation

$$\det J_{H(\cdot; y_2)}(z_1) = \det \left( \frac{dH(z_1; y_2)}{dz_1} \right) = \det \left( \frac{dS_1(z_1; R^{-1}(y_2; z_1))}{dz_1} \right). \quad (12)$$

We have to evaluate the total derivative w.r.t. the components of  $z_1$ . In the rightmost matrix, we have two contributions, coming from derivations acting on the first and second arguments, namely

$$\begin{aligned} \frac{dS_1(z_1; R^{-1}(y_2; z_1))}{dz_1} &= \left( \frac{dS_1(z_1; R^{-1}(y_2; u_1))}{dz_1} + \frac{dS_1(u_1; R^{-1}(y_2; z_1))}{dz_1} \right) \Big|_{u_1=z_1} \\ &= \left( \frac{dS_1(z_1; z_2)}{dz_1} + \frac{dR^{-1}(y_2; z_1)}{dz_1} \frac{dS_1(z_1; z_2)}{dz_2} \right) \end{aligned} \quad (13)$$

and finally recognize that (here  $I_n$  is the  $n$ -dimensional identity matrix)

$$0 = \frac{dI_{n_2}}{dz_1} = \frac{dR(R^{-1}(y_2; z_1); z_1)}{dz_1} = \frac{dR(z_2; z_1)}{dz_1} + \frac{dR^{-1}(y_2; z_1)}{dz_1} \frac{dR(z_2; z_1)}{dz_2} \quad (14)$$

from which

$$\frac{dR^{-1}(y_2; z_1)}{dz_1} = -\frac{dR(z_2; z_1)}{dz_1} \left( \frac{dR(z_2; z_1)}{dz_2} \right)^{-1} = -\frac{dS_2(z_1, z_2)}{dz_1} (J_{R(\cdot; z_1)})^{-1}. \quad (15)$$

Substituting (15) into (13), and comparing to (??), leads to the conclusion.  $\square$

**Lemma 2.2** (Partial elimination, invertible version). *Let  $S$ ,  $R$  and  $H$  as in Lemma 2.1. In particular, assume by hypothesis that  $R(\cdot; z_1) \in \mathcal{J}_{n_2}$  for all  $z_1 \in \mathbb{K}^{n_1}$ . We have*

$$S(z) \in \mathcal{J}_{N; n_1} \quad \text{iff} \quad H(\cdot; 0) \in \mathcal{J}_{n_1}. \quad (16)$$

*Proof.* We shall prove that  $S^{-1}(\cdot; 0) \in \mathcal{P}_{n_1}$  exists if and only if  $H^{-1}(\cdot; 0) \in \mathcal{P}_{n_1}$  exists.

The direction “ $H \Rightarrow S$ ” is obvious: just take  $S^{-1}$  as

$$(S^{-1})_1(y_1, y_2) = H^{-1}(y_1; y_2); \quad (S^{-1})_2(y_1, y_2) = R^{-1}(y_2; H^{-1}(y_1; y_2)). \quad (17)$$

For the opposite direction, “ $S \Rightarrow H$ ”, we start by assuming to have

$$z_1 = (S^{-1})_1(y_1, y_2); \quad z_2 = (S^{-1})_2(y_1, y_2); \quad (18)$$

but we already know that  $z_2 = R^{-1}(y_2; z_1)$ . Thus, from the unicity of the inverse, we obtain that

$$(S^{-1})_2(y_1, y_2) = R^{-1}(y_2; (S^{-1})_1(y_1, y_2)). \quad (19)$$

From the definition of  $H$  in terms of  $S$ , we get

$$H(z_1; y_2) = S_1(z_1, R^{-1}(y_2; z_1)). \quad (20)$$

Calculate

$$\begin{aligned} H((S^{-1})_1(y_1, y_2); y_2) &= S_1((S^{-1})_1(y_1, y_2), R^{-1}(y_2; (S^{-1})_1(y_1, y_2))) \\ &= S_1((S^{-1})_1(y_1, y_2), (S^{-1})_2(y_1, y_2)) = S_1(S^{-1}(y_1, y_2)) = y_1; \end{aligned} \quad (21)$$

where we used Equation (19). From the unicity of the inverse (when it exists), and its characterizing equation  $H(H^{-1}(y_1; y_2); y_2) = y_1$ , we can identify

$$H^{-1}(y_1; y_2) = (S^{-1})_1(y_1, y_2) \quad (22)$$

and thus conclude. □

As announced above, we now prove Theorem 1.6.

*Proof.* Let us outline the proof. From  $F \in \mathcal{P}_{n,d+1}$ , we will construct a function  $\tilde{F} = \Phi(F) \in \mathcal{P}_{n(n+1),d}$  with  $F(z^{(1)}) = H(z^{(1)}; 0)$ . Using Lemma 2.1, if  $F \in \mathcal{J}_{n,d+1}^{\text{lin}}$ , we then have that  $\tilde{F} \in \mathcal{J}_{n(n+1),d,n}^{\text{lin}}$ , so

$$\Phi(\mathcal{J}_{n,d+1}^{\text{lin}}) = \mathcal{J}_{n(n+1),d;n}^{\text{lin}} \cap \text{Im}(\Phi).$$

By using Lemma 2.2 in a similar way, we also have  $\Phi(\mathcal{J}_{n,d+1}) = \mathcal{J}_{n(n+1),d;n} \cap \text{Im}(\Phi)$ .

We start from  $F(z^{(1)}) \in \mathcal{P}_{n,d+1}$ . Trivial arguments allow to establish that  $F \in \mathcal{J}_n$  iff  $F - F(0) \in \mathcal{J}_n$ , i.e. we can drop the part of degree zero in  $F$  (see e.g. [BCW82, Prop. 1.1, pag. 303]). Thus a generic  $F \in \mathcal{P}_{n,d+1}$  has the form

$$F(z^{(1)}) = \sum_{c=1}^{d+1} F_c(z^{(1)}); \quad (23)$$

where  $F_c$  is homogeneous of degree  $c$ .

From  $F \in \mathcal{P}_{n,d+1}$ , we will construct a  $\tilde{F} = \Phi(F) \in \mathcal{P}_{n(n+1),d}$  such that we can identify  $F(z^{(1)}) = H(z^{(1)}; 0)$ , where  $H(z^{(1)}; y^{(2)})$  is associated to  $\tilde{F}$

in the way this is done in Lemma 2.1. We indeed use here notations similar to those of Lemma 2.1, with  $S = \tilde{F}$  and  $N = n(n+1)$ , except that, as we use explicit component indices, we use upper-scripts for blocks (e.g.  $\tilde{F}^{(1)}(z^{(1)}, z^{(2)})$ , instead of  $F_1(z_1, z_2)$ , and  $z^{(1)} = \{z_i^{(1)}\}_{1 \leq i \leq n}$ ). Clearly we have  $n_1 = n$  and  $n_2 = n^2$ . As our construction is structured, we use double indices for components in the second block, i.e.  $z^{(2)} = \{z_{ij}^{(2)}\}_{1 \leq i, j \leq n}$ , instead of  $z^{(2)} = \{z_\ell^{(2)}\}_{1 \leq \ell \leq n^2}$ .

Set now  $\tilde{F}$  of degree at most  $d$  and block dimensions  $n$  and  $n^2$ , with explicit expression

$$\tilde{F}_i^{(1)}(z^{(1)}, z^{(2)}) = \sum_j z_{ij}^{(2)} z_j^{(1)}; \quad (24)$$

$$\tilde{F}_{ij}^{(2)}(z^{(1)}, z^{(2)}) = z_{ij}^{(2)} - \sum_c \frac{1}{c} \frac{d}{dz_j^{(1)}} (F_c)_i(z^{(1)}). \quad (25)$$

The complicated rightmost summand in  $\tilde{F}_{ij}^{(2)}$  only depends on  $z^{(1)}$ , so that in fact  $R(z^{(2)}; z^{(1)})$  is linear, and its invertibility is trivially established, namely

$$R^{-1}(y^{(2)}; z^{(1)}) = y_{ij}^{(2)} + \sum_c \frac{1}{c} \frac{d}{dz_j^{(1)}} (F_c)_i(z^{(1)}). \quad (26)$$

We only have to check that  $H$ , specialized to  $y^{(2)} = 0$ , is equal to  $F$ , i.e. that

$$F(z^{(1)}) = H(z^{(1)}; 0) := \tilde{F}^{(1)}(z^{(1)}, R^{-1}(0; z^{(1)})). \quad (27)$$

Dropping the now useless superscripts this reads

$$\begin{aligned} (\tilde{F}^{(1)})_i(z, R^{-1}(0; z)) &= \sum_j z_j \left( 0 + \sum_c \frac{1}{c} \frac{d}{dz_j} (F_c)_i(z) \right) \\ &= \sum_c \frac{1}{c} \sum_j z_j \frac{d}{dz_j} (F_c)_i(z) = F_i(z), \end{aligned} \quad (28)$$

as was to be proven.<sup>2</sup>

□

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<sup>2</sup>We used the obvious fact that if  $A(x_1, \dots, x_n)$  is a homogeneous polynomial of degree  $d$ , one has

$$\frac{1}{d} \sum_{i=1}^n x_i \frac{d}{dx_i} A(x) = A(x). \quad (29)$$



### 3 QFT proof of the reduction theorem

#### 3.1 Heuristic QFT proof

We consider in this section QFT arguments which are *heuristic*, as they involve rewritings of the involved algebraic quantities, in terms of formal integrals, deformations of Gaussian integrals, that do not generally converge. The reasonings implying (some of) the facts that can be derived by these methods are illustrated in [Abd03b]. Although, in our case, it would just be easier to translate our procedure, step by step, into a purely algebraic one. We did not perform this here, as we find that the QFT formalism provides a notational shortcut and a useful visualization of the algebraic derivation.

Let us start by briefly recalling what we shall call the *Abdesselam–Rivasseau model* (see [Abd03a] for details). This model is a ‘zero-dimensional QFT model’. Here, the ‘dimension’  $D$  refers to the fact that QFT models are stated in terms of functional integrals, for field  $\phi_i(x)$  depending on a discrete index  $i$  and a continuous coordinate  $x \in \mathbb{R}^D$ . Here we are in the much simpler case  $D = 0$ , i.e. we have only discrete indices, this fact being in part responsible for the possibility of producing rigorous proofs within this formalism (in such a situation, some authors refer to a *combinatorial QFT*). Note that  $D$  shall not be confused with the dimension  $n$  of the linear system  $F(z)$ . We anticipate that our fields will be complex variables, in holomorphic basis, and the associated integrals will be on  $\mathbb{C}^n$  (i.e., with measure  $d\phi d\phi^\dagger$ ).

Now, let  $n, d \geq 1$ , and let  $F \in \mathcal{P}_{n,d}$ . Invertibility of  $F_1(z)$  is equivalent to the invertibility of  $F_2(z) := F_1(Rz+u)$ , for  $R \in \text{GL}(n, \mathbb{C})$  and  $u \in \mathbb{C}^n$ , and  $F_1, F_2$  have the same degree, thus w.l.o.g. we can assume that  $F(z) = z + \mathcal{O}(z^2)$ . In such a case the coordinate functions of  $F$  can be written as

$$F_i(z) = z_i - \sum_{k=2}^d \sum_{j_1, \dots, j_k=1}^n w_{i,j_1 \dots j_k}^{(k)} z_{j_1} \dots z_{j_k} =: z_i - \sum_{k=2}^d W_i^{(k)}(z),$$

for  $i \leq n$  and  $w_{i,j_1 \dots j_k}^{(k)}$  some coefficients.<sup>3</sup> We introduce the inhomogeneous extension of the QFT model of [Abd03a].

$$Z(J, K) = \int_{\mathbb{C}^n} d\varphi d\varphi^\dagger e^{-\varphi^\dagger \varphi + \varphi^\dagger \sum_{k=2}^d W^{(k)}(\varphi) + J^\dagger \varphi + \varphi^\dagger K},$$

where  $J, K$  are vectors in  $\mathbb{C}^n$ . The full expression is called *partition function*, the expression in the exponential is called *action*, the coefficients  $w$  are called

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<sup>3</sup>Although only the symmetrized quantities  $\sum_{\sigma \in \mathfrak{S}_k} w_{i,j_{\sigma(1)} \dots j_{\sigma(k)}}^{(k)}$  contribute, it is convenient to keep this redundant notation.

*coupling constants*, while  $J$  and  $K$  are called *external sources*. When the coupling constants are set to zero, the integral is calculated by Gaussian integration:

$$\int_{\mathbb{C}^n} d\varphi d\varphi^\dagger e^{-\varphi^\dagger \varphi + J^\dagger \varphi + \varphi^\dagger K} = e^{J^\dagger K}. \quad (30)$$

We can then express the unique formal inverse  $G$  of  $F$ . Indeed, for  $H_i$  an analytic function and  $u \in \mathbb{C}^n$ ,

$$\begin{aligned} \int d\varphi d\varphi^\dagger H_i(\varphi) e^{-\varphi^\dagger F(\varphi) + \varphi^\dagger u} &= \int d\tilde{\varphi} d\varphi^\dagger H_i(G(\tilde{\varphi} + u)) e^{-\varphi^\dagger \tilde{\varphi}} \det(\partial G(\tilde{\varphi} + u)) \\ &= \int d\tilde{\varphi} H_i(G(\tilde{\varphi} + u)) \delta(\tilde{\varphi}) \det(\partial G(\tilde{\varphi} + u)) = H_i(G(u)) \det(\partial G(u)), \end{aligned}$$

with the change of variables:  $\tilde{\varphi} = F(\varphi) - u$ . Taking the ratio of such expressions, for  $H_i(z) = z_i$  at numerator, and  $H_i(z) = 1$  at denominator, we obtain that the formal inverse corresponds to the one-point (outgoing) correlation function:

$$G_i(u) = \frac{\int_{\mathbb{C}^n} d\varphi d\varphi^\dagger \varphi_i e^{-\varphi^\dagger \varphi + \varphi^\dagger \sum_{k=2}^d W^{(k)}(z) + \varphi^\dagger u}}{\int_{\mathbb{C}^n} d\varphi d\varphi^\dagger e^{-\varphi^\dagger \varphi + \varphi^\dagger \sum_{k=2}^d W^{(k)}(z) + \varphi^\dagger u}} \quad (31)$$

Moreover, the partition function coincides with the inverse of the Jacobian:

$$Z(0, u) = \det(\partial G(u)) = JG(u) = \frac{1}{JF(G(u))}.$$

The sets of polynomial functions involved in the Jacobian Conjecture can be rephrased in this framework:

$$\begin{aligned} \mathcal{J}_{n,d}^{\text{lin}} &= \{F \in \mathcal{P}_{n,d} \mid Z(0, u) = 1 \forall u \in \mathbb{C}^n\}, \\ \mathcal{J}_{n,d} &= \{F \in \mathcal{P}_{n,d} \mid G_i(u) \text{ given by (31) is in } \mathcal{P}_n\}. \end{aligned}$$

Let us now introduce the *intermediate field method* to reduce the degree  $d$  of  $F$ . We will thus add  $n^2$  “intermediate fields”  $\sigma$  to the model. Indeed, we have, from the general formula (30) of Gaussian integration,

$$\begin{aligned} e^{(\varphi_i^\dagger \varphi_j) \left( \sum_{j_2, \dots, j_d=1}^n w_{i,j,j_2 \dots j_d}^{(d)} \varphi_{j_2} \dots \varphi_{j_d} \right)} \\ = \int_{\mathbb{C}^{n^2}} d\sigma_{i,j} d\sigma_{i,j}^\dagger e^{-\sigma_{i,j}^\dagger \sigma_{i,j} + \sigma_{i,j}^\dagger \left( \sum_{j_2, \dots, j_d=1}^n w_{i,j,j_2 \dots j_d}^{(d)} \varphi_{j_2} \dots \varphi_{j_d} \right) + (\varphi_i^\dagger \varphi_j) \sigma_{i,j}} \quad (32) \end{aligned}$$

We now use the identity (32), for each pair  $(i, j)$ , in the partition function of the model with  $n$  dimensions and degree  $d$ , in order to re-express the monomials of degree  $d$  in the fields  $\varphi$ . This leads to:

$$Z(J, K) = \int_{\mathbb{C}^n} d\varphi d\varphi^\dagger \int_{\mathbb{C}^{n^2}} d\sigma d\sigma^\dagger e^{-\varphi^\dagger \varphi + \varphi^\dagger \sum_{k=2}^{d-1} W^{(k)}(\varphi) + J^\dagger \varphi + \varphi^\dagger K} e^{\sum_{i,j=1}^n \left( -\sigma_{i,j}^\dagger \sigma_{i,j} + \sigma_{i,j}^\dagger \sum_{j_2, \dots, j_d=1}^n w_{i,j,j_2 \dots j_d}^{(d)} \varphi_{j_2} \dots \varphi_{j_d} + \varphi_i^\dagger \varphi_j \sigma_{i,j} \right)}.$$

We define the new vector  $\phi$  of  $\mathbb{C}^{n+n^2}$  by  $\phi = (\varphi_1, \dots, \varphi_n, \sigma_{1,1}, \dots, \sigma_{1,n}, \dots, \sigma_{n,1}, \dots, \sigma_{n,n})$ . We further define the interaction coupling constants  $\tilde{w}$  as:

- for  $k = d - 1$ , we set  $\tilde{w}_{i,j,j_2 \dots j_d}^{(d-1)} := w_{i,j,j_2 \dots j_d}^{(d-1)}$  and  $\tilde{w}_{i,n+j,j_2 \dots j_d}^{(d-1)} = w_{i,j,j_2 \dots j_d}^{(d)}$  with  $i, j, j_2, \dots, j_n \leq n$
- for  $k \in \{3, \dots, d - 2\}$ , we set  $\tilde{w}_{i,j,j_2 \dots j_k}^{(k)} := w_{i,j,j_2 \dots j_k}^{(k)}$  with  $i, j, j_2, \dots, j_n \leq n$
- for  $k = 2$ , we set  $\tilde{w}_{i,j,j_2}^{(2)} := w_{i,j,j_2}^{(2)}$  and  $\tilde{w}_{i,j,i+n+j}^{(2)} = 1$  with  $i, j, j_2 \leq n$ .

The remaining coefficients of  $\tilde{w}$  are set to 0.

In the same way, the external sources are defined to be  $\tilde{J} = (J, 0)$  and  $\tilde{K} = (K, 0)$ , where, of course, the number of extra vanishing coordinates is  $n^2$ . It is important to note that these external sources have fewer degrees of freedom than coordinates ( $n$  vs.  $n(n+1)$ ). We also remark that, for generic  $d$ , in order to have a relation adapted to induction, it is crucial to consider an inhomogeneous model, since the intermediate field method originates terms of degrees  $d - 1$  and 3.

One now has

$$Z(J, K) = \int_{\mathbb{C}^{n+n^2}} d\phi d\phi^\dagger e^{-\phi^\dagger \phi + \phi^\dagger \sum_{k=2}^{d-1} \tilde{W}^{(k)}(\phi) + \tilde{J}^\dagger \phi + \phi^\dagger \tilde{K}}$$

and

$$G_i(u) = \frac{\int_{\mathbb{C}^{n+n^2}} d\phi d\phi^\dagger \phi_i e^{-\phi^\dagger \phi + \phi^\dagger \sum_{k=2}^{d-1} \tilde{W}^{(k)}(\phi) + \phi^\dagger \tilde{u}}}{\int_{\mathbb{C}^{n+n^2}} d\phi d\phi^\dagger e^{-\phi^\dagger \phi + \phi^\dagger \sum_{k=2}^{d-1} \tilde{W}^{(k)}(\phi) + \phi^\dagger \tilde{u}}},$$

for  $i \in \{1, \dots, n\}$ .

We have thus showed in a heuristic way that the partition function (resp. the one-point correlation function) of the model with dimension  $n \in \mathbb{N}$  and degree  $d \in \mathbb{N} \setminus \{1, 2\}$  is equal to the partition function (resp. the  $n$  first coordinates of the one-point correlation function) of the model with dimension  $n(n+1)$  and degree  $d - 1$ , up to a redefinition of the coupling constant  $w \mapsto \tilde{w}$

and a trivial redefinition of the external sources. Since the partition function corresponds to the inverse of the Jacobian (resp. the one-point correlation function corresponds to the formal inverse), this gives, as announced above, a heuristic proof of Theorem 1.6.

### 3.2 Formal inverse in QFT

In this section we adopt notations as above, but we proceed at a more formal level. In particular, the coefficients  $w$  are considered as formal indeterminates in (multi-dimensional) power series. In order to project more easily to the simpler context of univariate power series, we introduce a further, redundant, indeterminate  $\theta$ , by replacing each coefficient  $w_{i,j_1\dots j_k}^{(k)}$  by  $\theta^{k-1}w_{i,j_1\dots j_k}^{(k)}$ . We denote by  $\mathbb{C}[[\theta]]$  the ring of formal power series in  $\theta$ . The exponent of  $\theta$  in the  $w$ 's measures the “spin” of the associated monomial, i.e., the action is invariant under the transformation  $\phi_j \rightarrow \phi_j e^{i\omega}$ ,  $\phi_j^\dagger \rightarrow \phi_j^\dagger e^{-i\omega}$ ,  $\theta \rightarrow \theta e^{-i\omega}$ .

The polynomial function  $F$  now is extended naturally to a function from  $\mathbb{C}[[\theta]]^n$  to itself, although we are ultimately interested on invertibility on  $\mathbb{C}^n$ . The integrals of the previous subsection are now well defined as a formal expansion in  $\theta$

$$Z(J, K) := \int_{\mathbb{C}^n} d\varphi d\varphi^\dagger e^{-\varphi^\dagger \varphi + J^\dagger \varphi + \varphi^\dagger K} \sum_{r=0}^{\infty} \frac{1}{r!} \left( \varphi^\dagger \sum_{k=2}^d (\theta^{k-1} W^{(k)}(\varphi)) \right)^r, \quad (33)$$

where, as in the previous section,  $J$  and  $K$  can be considered as vectors in  $\mathbb{C}^n$  (as they enter the quadratic part of the action, there is no need of promoting them to formal indeterminates).

Any term of this power series, i.e.  $[\theta^r]Z(J, K)$  for  $p \in \mathbb{N}$ , can be calculated as a finite linear combination of terms of the following form, with  $r = q - p$

$$\int_{\mathbb{C}^n} d\varphi d\varphi^\dagger e^{-\varphi^\dagger \varphi + J^\dagger \varphi + \varphi^\dagger K} \varphi_{i_1}^\dagger \dots \varphi_{i_p}^\dagger \varphi_{j_1} \dots \varphi_{j_q} \quad (34)$$

which, of course, is also given by

$$\frac{\partial^p}{\partial K_{i_1} \dots \partial K_{i_p}} \frac{\partial^q}{\partial J_{j_1}^\dagger \dots \partial J_{j_q}^\dagger} \int_{\mathbb{C}^n} d\varphi d\varphi^\dagger e^{-\varphi^\dagger \varphi + J^\dagger \varphi + \varphi^\dagger K} \quad (35)$$

An analysis of such an expression, in light of (30), leads to the *Wick Theorem* (see for example textbooks such as [CM08]): the integral in (34) is equal to the sum over all possible substitutions, in the monomial  $\varphi_{i_1}^\dagger \dots \varphi_{i_p}^\dagger \varphi_{j_1} \dots \varphi_{j_q}$  of the patterns  $\varphi_i^\dagger \rightarrow J_i^\dagger$ ,  $\varphi_j \rightarrow K_j$ , and  $\varphi_i^\dagger \varphi_j \rightarrow \delta_{ij}$ , up to have no  $\varphi$ 's and  $\varphi^\dagger$ 's left.

One can then associate graphical representations  $\Gamma$  to these expressions, going under the name of *Feynman graphs*. For the model analyzed here, the set of graphs and their associated weights are obtained through the following rules:

- vertices in the graph have indices  $i \in \{1, \dots, n\}$  attached to the incident edges;
- a term  $\delta_{ij}$  is represented as a directed edge, with index  $i$  at its endpoints
- a term  $J_i^\dagger$  is represented as a vertex of in-degree 1 and out-degree 0, incident to an edge of index  $i$ ;
- similarly, a term  $K_j$  is represented as a vertex of in-degree 0 and out-degree 1, incident to an edge of index  $j$ ;
- a weight  $\theta^{k-1} w_{i,j_1 \dots j_k}^{(k)}$  is associated to a vertex with out-degree 1 (and index  $i$ ), and in-degree  $k$  (and indices  $\{j_1, \dots, j_k\}$ ). The incident edges have a cyclic ordering;
- a symmetry factor  $1/|\text{Aut}(\Gamma)|$  appears overall, as a combination of the  $1/r!$  factor and of multiple counting for the same diagram in the expansion of (33).

Note that, in general, several but finitely many graphs contribute to a given expression associated to a monomial  $\varphi_{i_1}^\dagger \dots \varphi_{i_p}^\dagger \varphi_{j_1} \dots \varphi_{j_q}$ .

For the problem at hand here, both when evaluating the Jacobian and the formal inverse of a component, we can restrict to the case  $J = 0$ . In order to conform to notations in the literature, we shall also rename  $K_j$ 's into  $u_j$ 's (indeed, when  $J = 0$ , the source  $K$  induces a formal translation of the components  $\phi$ 's, without translating the  $\phi^\dagger$ 's). It is easy to see that, in a theory with such a constrained set of vertex out-degrees, all contributing Feynman graphs with no  $J$ -leaves contain exactly one cycle<sup>4</sup> per connected component (see Fig. 1), while connected graphs with a unique  $J$ -leaf correspond to directed trees rooted at this leaf.

We can now state and prove the expression for the formal inverse, which in fact coincides with the heuristically-derived (31), as well as the expression of the partition function (33). Both have been already derived in [Abd03a].

**Theorem 3.1.** *Define  $\mathcal{A}_i(T)(u)$  the amplitude of a tree  $T$  with exactly one outgoing edge, of index  $i$ . The formal inverse of the function  $F$  is the function*

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<sup>4</sup>Cycles are commonly called “loops” in QFT.

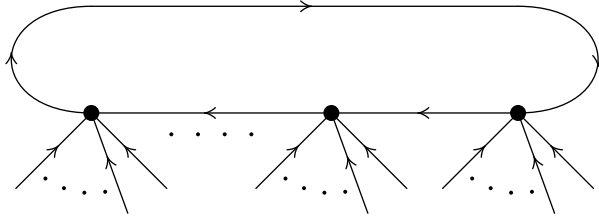


Figure 1: Diagrammatic representation of a Feynman graph with no external outgoing edges, thus having one loop.

$G$  with coordinates

$$G_i(u) = \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{T:|T|=r} \mathcal{A}_i(T)(u), \quad (36)$$

where  $T$  denotes a tree with one outgoing edge, and  $|T|$  is the number of vertices in  $T$ .

Note an abuse of notation here, as  $i$  is not an index pertinent to the function  $\mathcal{A}$ , but to the variable  $T$ . We can equivalently think that  $T$  has its root unlabeled (and the vertex adjacent to the root comes with no weight), and the function  $\mathcal{A}_i$  completes the weight of  $T$  by including the appropriate factor  $w_{i,j_1 \dots j_k}^{(k)}$ .

*Proof.* The proof is a straightforward adaptation of [Abd03a] to the inhomogeneous case. If the tree  $T$  has at least one vertex, its Feynman integral can be re-expressed as

$$\mathcal{A}_i(T) = \sum_{k=2}^d \theta^{k-1} \sum_{i_1, \dots, i_k=1}^n w_{i, i_1 \dots i_k}^{(k)} \mathcal{A}_{i_1}(T_1) \dots \mathcal{A}_{i_k}(T_k),$$

where we have denoted by  $T_1, \dots, T_k$  the subtrees with one outgoing external edge which is adjacent in  $T$  to the vertex linked to the external edge  $i$ . If  $r_1, \dots, r_k$  denote the sizes of these subtrees, we must have  $r_1 + \dots + r_k = r - 1$ . In this sum, one has to take into consideration the aforementioned symmetry factors. However, the automorphism group of a rooted tree is trivially expressed in terms of those of its branches, and the labels of the branches. We have a further multinomial factor  $\frac{r!}{1! r_1! \dots r_k!}$  for the total ordering of the vertices. Since  $G_i(u)$  is the sum of Feynman integrals of graphs with



responding to the choice of the first vertex in an oriented cycle. Note that

$$\sum_{\ell_1=1}^{k_1} \sum_{j_{(1,1)}, \dots, j_{(1,k_1)}=1}^n w_{j_{(r,\ell_r)}, j_{(1,1)}, j_{(1,2)}, \dots, j_{(1,k_1)}}^{(k_1)} G_{j_{(1,1)}}(u) G_{j_{(1,2)}}(u) \dots \overset{j_{(1,\ell_1)}}{\vee} \dots G_{j_{(1,k_1)}}(u) \\ = \sum_{i,j=1}^n \frac{\partial W_i^{(k_1)}(z)}{\partial z_j} \Big|_{z=G(u)},$$

so that, omitting the now trivial factor  $\theta$ , the whole contribution corresponds to:

$$\frac{1}{r} \sum_{i_1, \dots, i_r=1}^n \left( \delta_{i_1 i_2} - J(F)_{i_1 i_2}(G(u)) \right) \left( \delta_{i_2 i_3} - J(F)_{i_2 i_3}(G(u)) \right) \dots \left( \delta_{i_r i_1} - J(F)_{i_r i_1}(G(u)) \right).$$

One then has

$$\ln(Z(0, u)) = \sum_{r=1}^{\infty} \frac{1}{r} \text{Tr} \left( ((\mathbb{1} - J(F)(G(u)))^r \right) = -\text{Tr} \left( \ln(J(F)(G(u))) \right), \quad (38)$$

which concludes the proof.  $\square$

### 3.3 Proof of the theorem

We give in this section the combinatorial QFT proof of our main result, Theorem 1.6.

*Proof.* Consider a directed tree  $T$ , constructed from the Feynman rules defined in section 3.2, in dimension  $n$  (i.e. with edge indices in  $\{1, \dots, n\}$ ), and degree  $d$  (i.e. with vertices of in-degree at most  $d$ ). From Theorem 3.1, we know that  $\mathcal{A}_i(T)(u)$  is used to compute the formal inverses  $G_i(u)$ .

Let us now define the Feynman rules for the model in dimension  $n(n+1)$ , obtained using the intermediate field method described in Section 3.1. For  $i, j \in \{1, \dots, n(n+1)\}$ , one has:

- propagators, i.e. directed edges with index  $i$  correspond to the term  $\delta_{ij}$  (obtained by the  $\phi_i^\dagger \phi_j$  substitution in Wick Theorem);
- leaves with in-degree 1 correspond to the term  $\tilde{u}_j$ , and in particular, as  $\tilde{u}_j = 0$  for  $j > n$ , contributing diagrams have all leaf-indices in the range  $\{1, \dots, n\}$ ;
- vertices of coordination  $k+1$ , for  $k \in \{1, \dots, d-1\}$ , with one outgoing edge of index  $i$ , and  $k$  incoming edges of indices  $j_1, \dots, j_k$ , correspond to the term  $\theta^{k-1} \tilde{w}_{i, j_1 \dots j_k}^{(k)}$ . Indices of the vertices are summed on.



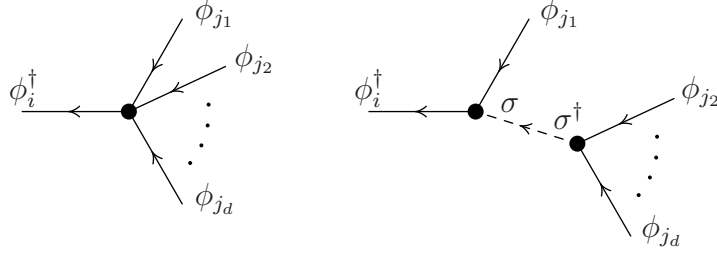


Figure 2: Diagrammatic representation of the intermediate field method.

For a graphical representation of the intermediate field method leading to the model in dimension  $n(n+1)$ , see Figure 2. As stated in Section 3.1,  $\tilde{u} = (u_1, \dots, u_n, 0, \dots, 0)$ , and the coefficients  $\tilde{w}$  are

- for  $k = d-1$ , set  $\tilde{w}_{i,j,j_2 \dots j_d}^{(d-1)} := w_{i,j,j_2 \dots j_d}^{(d-1)}$ ; and  $\tilde{w}_{i,n+j,j_2 \dots j_d}^{(d-1)} = w_{i,j,j_2 \dots j_d}^{(d)}$  with  $i, j, j_2, \dots, j_n \leq n$ .
- for  $k \in \{3, \dots, d-2\}$ , set  $\tilde{w}_{i,j,j_2 \dots j_k}^{(k)} := w_{i,j,j_2 \dots j_k}^{(k)}$  with  $i, j, j_2, \dots, j_n \leq n$ .
- for  $k = 2$ , set  $\tilde{w}_{i,j,j_2}^{(2)} := w_{i,j,j_2}^{(2)} = 0$ ; and  $\tilde{w}_{i,j,i+n+j}^{(2)} = 1$  with  $i, j, j_2 \leq n$ .

The other components of  $\tilde{w}$  are set to 0 by definition. Note that this corresponds to consider the new polynomial map  $\tilde{F} : \mathbb{C}^{n(n+1)} \rightarrow \mathbb{C}^{n(n+1)}$  given by

$$\begin{aligned} \tilde{F}_i(z) &= z_i - \sum_{k=3}^{d-1} \sum_{j_1, \dots, j_k=1}^n w_{i,j_1 \dots j_k}^{(k)} z_{j_1} \dots z_{j_k} - \sum_{j=1}^n z_j z_{i+n+j} \quad \text{for } 1 \leq i \leq n, \\ \tilde{F}_{i+n+j}(z) &= z_{i+n+j} - \sum_{j_2, \dots, j_d=1}^n w_{i,j,j_2 \dots j_d}^{(d)} z_{j_2} \dots z_{j_d} \quad \text{for } 1 \leq i, j \leq n. \end{aligned}$$

To each tree  $T$  in the theory of dimension  $n$ , and a choice of incoming edge per each vertex of degree  $d+1$ , we can associate canonically a tree  $\tilde{T}$ , constructed from these new Feynman rules. Propagators, leaves and vertices of coordination less than or equal to  $d$  in the tree  $T$  are identically transposed in the tree  $\tilde{T}$ , while a vertex in  $T$  of coordination  $d+1$  with one outgoing edge  $i$  and  $d$  incoming edges of indices  $j_1, \dots, j_d$  is split into two vertices in  $\tilde{T}$ , connected by an edge of index  $n < i \leq n(n+1)$ . Edges of such indices are never adjacent on the tree. The precise construction is depicted in Figure 2.

Consider now the Feynman integral  $\tilde{\mathcal{A}}_i(\tilde{T})(\tilde{u})$ , in the model of dimension  $n(n+1)$ . For propagators, leaves and vertices of coordination less than or equal to  $d$  in the tree  $T$ , the contribution to  $\tilde{\mathcal{A}}_i(\tilde{T})(\tilde{u})$  is the same as

in  $\mathcal{A}_i(T)(u)$ , because the summation in  $i, j_1, \dots, j_k$  of the vertices reduce to  $\{1, \dots, n\}$ , except for  $k = d - 1$  where  $i$  could a priori take value in  $\{n + 1, \dots, n(n + 1)\}$ . However, the outgoing edge of this vertex is either the external edge (with  $i \in \{1, \dots, n\}$ ) or is adjacent to another vertex (of coordination greater than or equal to four by hypothesis), so we also have  $i \in \{1, \dots, n\}$ .

For any vertex of coordination  $d + 1$  in  $T$ , the above coefficients  $\tilde{w}_{i \cdot n + j, j_2 \dots j_d}^{(d-1)}$  and  $\tilde{w}_{i, j, i \cdot n + j}^{(2)}$  have been chosen so that the contribution in  $\mathcal{A}_i(T)(u)$  coincides with the one in  $\tilde{\mathcal{A}}_i(\tilde{T})(\tilde{u})$ . The only thing to check is that the index  $\ell$  relating the two new vertices in  $\tilde{T}$  is summed over only  $\ell \in \{n + 1, \dots, n(n + 1)\}$ . This is the case because  $\tilde{w}_{i, j, \ell}^{(2)} = 0$  for  $\ell \leq n$ . Note also that the formal factor  $\theta^{d-1}$  for the vertex in  $T$  corresponds to  $\theta$  for the new vertex of coordination 3 and  $\theta^{d-2}$  for the one of coordination  $d$  in  $\tilde{T}$ . Due to Feynman rules, only trees  $\tilde{T}$  obtained from a tree  $T$  can contribute to the formal inverse  $\tilde{G}_i(\tilde{u})$  of Theorem 3.1.

One then concludes that

$$\tilde{\mathcal{A}}_i(\tilde{T})(\tilde{u}) = \mathcal{A}_i(T)(u). \quad (39)$$

Due to Theorem 3.1, we have  $\tilde{G}_i(\tilde{u}) = G_i(u)$  for  $i \leq n$ . So we can associate injectively to a polynomial function  $F \in \mathcal{P}_{n, d}$  another function  $\tilde{F} = \Phi(F) \in \mathcal{P}_{n(n+1), d-1}$  and we proved that  $F \in \mathcal{J}_{n, d} \Leftrightarrow \tilde{F} \in \mathcal{J}_{n(n+1), d-1; n}$ .

The same process can be performed for graphs without external edges, which leads to the equality of partition functions in both QFT models, the one of dimension  $n(n + 1)$  and the one of dimension  $n$ :

$$\tilde{Z}(0, \tilde{u}) = Z(0, u). \quad (40)$$

Moreover, for the one point correlation function with one external leg in the auxiliary field, equation (37) leads to the following expression:

$$\tilde{G}_{i \cdot n + j}(\tilde{u}) = \theta^{d-2} \sum_{j_2, \dots, j_d} w_{i, j, j_2 \dots j_d}^{(d)} G_{j_2}(u) \dots G_{j_d}(u) = R_{i \cdot n + j}^{-1}(0, \theta G(u)),$$

since  $R_{i \cdot n + j}(v, u) = v_{i \cdot n + j} - \sum_{j_2, \dots, j_d} w_{i, j, j_2 \dots j_d}^{(d)} u_{j_2} \dots u_{j_d}$  (see Definition 1.5). By Proposition 3.3, we then obtain

$$\begin{aligned} \det(J_{\tilde{F}})(\theta G(u), R^{-1}(0, \theta G(u))) &= \det(J_{\tilde{F}})(\theta \tilde{G}(\tilde{u})) = \tilde{Z}(0, \tilde{u})^{-1} = Z(0, u)^{-1} \\ &= \det(J_F)(\theta G(u)). \end{aligned} \quad (41)$$

This proves the second part of the Theorem, namely  $F \in \mathcal{J}_{n, d}^{\text{lin}} \Leftrightarrow \tilde{F} \in \mathcal{J}_{n(n+1), d-1; n}^{\text{lin}}$ .  $\square$

## 4 Example

Let us illustrate the conjecture in low dimension  $n = 2$ ,  $n' = 1$  and arbitrary degree  $d$  in this section. This won't be useful for the Jacobian conjecture of course, because it shows the case  $n = 1$ ,  $d + 1$ , which is trivial. But it will give explicit computations involving the definitions introduced above.

We consider the polynomial given by

$$\begin{aligned} F_1(z) &= z_1 - \sum_{k=0}^d a_{1,k} z_1^k z_2^{d-k} \\ F_2(z) &= z_2 - \sum_{k=0}^d a_{2,k} z_1^k z_2^{d-k}, \end{aligned}$$

where the complex coefficients  $a_{i,k}$  are fixed. Then the Jacobian takes the form

$$\begin{aligned} \det(J_F)(z) &= 1 - \sum_{k=0}^{d-1} (a_{1,k+1}(k+1) + a_{2,k}(d-k)) z_1^k z_2^{d-1-k} \\ &\quad + \sum_{k,l=0}^d a_{1,k} a_{2,l} (d-k) l z_1^{k+l-1} z_2^{2d-k-l-1} \quad (42) \end{aligned}$$

In the standard case  $n' = n = 2$ , by using (42), the equation  $\det(J_F) = 1$  leads to the conditions

- For any  $k \in \{0, \dots, d-1\}$ ,  $a_{1,k+1}(k+1) = a_{2,k}(d-k)$ .
- For any  $m \geq 1$ ,  $\sum_{k=0}^{\min(d,m)} a_{1,k} a_{2,m-k} d(2k-m) = 0$ .

If  $F$  satisfies these conditions,  $F$  lies in  $\mathcal{J}_{2,d}^{\text{lin}}$ .

Let us describe the polynomials  $F$  that belong to  $\mathcal{J}_{2,d;1}^{\text{lin}}$  and compare with the above conditions. We will see that they are very different. Moreover, we will see that  $\mathcal{J}_{2,d;1}^{\text{lin}} = \mathcal{J}_{2,d;1}$ .

For  $n' = 1$ , we have to set

$$R(z_2; z_1) = z_2 - \sum_{k=0}^d a_{2,k} z_1^k z_2^{d-k}.$$

This expression has to be invertible as a polynomial in  $z_2$  and for any parameter  $z_1 \in \mathbb{C}$ . In particular, the Jacobian of  $R$  with respect to  $z_2$  has to be constant, which implies  $a_{2,k} = 0$  for any  $k < d-1$ . So  $R(z_2; z_1) =$

$z_2 - a_{2,d-1}z_1^{d-1}z_2 - a_{2,d}z_1^d$ . But the invertibility of  $R$  for any  $z_1$  also implies that  $a_{2,d-1} = 0$ .

Eventually, we get  $R(z_2; z_1) = z_2 - a_{2,d}z_1^d$ , and  $R^{-1}(y_2; z_1) = y_2 + a_{2,d}z_1^d$ . Let us look at the condition  $\det(J_F)(z_1, R^{-1}(0; z_1)) = 1$ . Replacing  $z_2$  by  $R^{-1}(0; z_1) = a_{2,d}z_1^d$  in (42), we find the following expression

$$\det(J_F)(z_1, R^{-1}(0; z_1)) = 1 + \sum_{k=1}^d a_{1,k}a_{2,d}^{d-k}(k(d-1) - d^2)z_1^{(d-1)(d-1-k)} + a_{1,0}a_{2,d}^d z_1^{(d-1)(d+1)}.$$

Then, the polynomial  $F$  belongs to  $\mathcal{J}_{2,d;1}^{\text{lin}}$  if and only if  $a_{1,d} = 0$  and

$$\forall k \in \{0, \dots, d-1\}, a_{1,k} = 0 \quad \text{or} \quad a_{2,d} = 0.$$

We see indeed that these conditions are very different from the one of  $\mathcal{J}_{2,d}^{\text{lin}}$ . Now, let us show that these polynomials  $F \in \mathcal{J}_{2,d;1}^{\text{lin}}$  are also in  $\mathcal{J}_{2,d;1}$ , so  $(F^{-1})_1(z_1, 0)$  is polynomial in  $z_1$ .

The first case of  $\mathcal{J}_{2,d;1}^{\text{lin}}$  corresponds to

$$F_1(z) = z_1, \quad F_2(z) = z_2 - a_{2,d}z_1^d.$$

Here, the global inverse is

$$F_1^{-1}(y) = y_1, \quad F_2(y) = y_2 + a_{2,d}y_1^d,$$

so the condition of  $\mathcal{J}_{2,d;1}$  is trivially satisfied. The second case coincides with polynomials

$$F_1(z) = z_1 - \sum_{k=0}^{d-1} a_{1,k}z_1^k z_2^{d-k}, \quad F_2(z) = z_2.$$

The global inverse is not polynomial in  $y_1, y_2$ . However, by setting  $u = (F^{-1})_1(y_1, 0)$ , we have the following equation  $y_1 = F_1(u, 0) = u$ , so

$$(F^{-1})_1(y_1, 0) = y_1$$

is polynomial in  $y_1$ , and  $F \in \mathcal{J}_{2,d;1}$ .

## 5 Concluding remarks and perspectives

We thus proved in this paper a reduction theorem to the quadratic case for the Jacobian conjecture, up to the addition of a new parameter  $n'$ . Moreover, we

did this first by using formal algebraic methods and then using QFT methods. This idea of using intermediate field method represents an illustration of how QFT methods can be successfully used to prove “purely” mathematical results.

Recall here that the Jacobian Conjecture is proved in the quadratic case by Wang in [Wan80]. The immediate perspective thus appears to be the adaptation of Wang’s proof to our particular case, where the parameter  $n'$  plays a non-trivial role. An interesting approach for this may be the reformulation of Wang’s proof in a QFT language, since we saw here that reduction results can be established in a natural way when using QFT techniques.

Let us end this paper by recalling that the Jacobian Conjecture is stably equivalent to the Dixmier Conjecture for endomorphisms of the Weyl algebra. It should be interesting to revisit the Dixmier Conjecture from the perspective of Noncommutative QFT (see [GW05, GMRT09, dGTW08, dG13] and references within) on the deformation quantization of the complex plane, which is an extension of the Weyl algebra (see [OMMY00, GdGvS14]).

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